The Electronic Journal of Mathematics and Technology, Volume 18, Number 2, ISSN 1933-2823

Uniform Convergence on Iterations generated by Special Convex Combinations of Parametric Equations

Wei-Chi YANG
wyang@radford.edu
Department of Mathematics
and Statistics
Radford University
Radford, VA 24142
USA

Abstract

This is an expansion and modification of the paper from [6]. We discuss the convergence of locus in the paper [5], which originated from a practice problem for the Chinese college entrance exam. Next, we extended some results in [6] from 2D to 3D. We are interested in the limit of a recursive sequence of loci that is built on a special convex combination of vectors involving curves or surfaces. We shall see many interesting graphs of uniform convergence of sequences generated by parametric curves and surfaces, which we hope to inspire many applications in computer graphics, and other related disciplines.

1 Introduction and Motivation

In [5], the problem is to find the locus that is determined by two fixed vectors using bisection theorem. In this paper, we discuss the proposed question of what will happen when we iterate the locus sequentially, and would like to find the limit of such locus. In short, we shall see a continuous deformation of an initial shape into a target shape, which is an interesting subject in computer graphics. We shall see the limit of a recursive sequence of convex combinations of vectors that involve curves or surfaces.

Original College Entrance Practice Problem: *Given a unit circle centered at* $(0,0)$ *and a fixed point at* $A = (2,0)$*. Let* Q *be a moving point on the unit circle* C*. Find the locus* M *which is the intersection between the angle bisector* QOA *and line segment* QA*.*

It is an easy exercise to verify that the locus of point M is a circle, which we leave as an exercise for the readers. Moreover, it is natural to imagine when DGS and CAS tools are available for students in a classroom as a project to explore, they may quickly pose 'what if' scenarios. We briefly state the following Exploratory Activity has been discussed in [4] and [5]. We then extend it to what we will focus on in this paper.

Exploratory Activity ([4] and [5])**:** Given an ellipse C: $[x(t), y(t)] = [a\cos(t), b\sin(t)], t \in [0, 2\pi]$, and a fixed point $A = (p, q) \notin C$. Let Q be a moving point on the ellipse (shown in green in Figure 1). Find the locus of the point M which is the intersection between the bisector QOA and line segment QA.

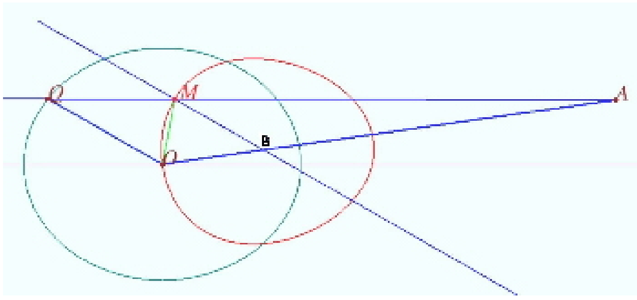


Figure 1. Locus, bisection and an ellipse

We derived that

$$\overrightarrow{OM} = \frac{OQ}{OA + OQ}\overrightarrow{OA} + \frac{OA}{OA + OQ}\overrightarrow{OQ}, \tag{1}$$

where $OQ = \left\|\overrightarrow{OQ}\right\| = \sqrt{a^2\cos^2 t + b^2\sin^2 t}$ and $OA = \left\|\overrightarrow{OA}\right\| = \sqrt{p^2 + q^2}$. We see that the parametric equation for the locus $M(t)$ can be plotted directly from Eq. (1) (see the red curve in Figure 1 above) with the help of a computational tool. It should cause no confusion throughout the paper that when $t \in [0, 2\pi]$, we often use $\overrightarrow{OM}$ to denote the vector $\overrightarrow{OM(t)}$, OQ stands for the magnitude of $\left\|\overrightarrow{OQ(t)}\right\|$ when $Q(t)$ is a parametric curve, and OA stands for the magnitude of $\left\|\overrightarrow{OA}\right\|$ if A is simply a point.

It is natural to extend our exploration and ask what would happen to the plot of

$$\overrightarrow{OM_{n+1}} = \begin{bmatrix} x_{n+1}(t) \\ y_{n+1}(t) \end{bmatrix} = \frac{OQ_n}{OA + OQ_n}\overrightarrow{OA} + \frac{OA}{OA + OQ_n}\overrightarrow{OQ_n}, \tag{2}$$

when $n \to \infty$, where $\overrightarrow{OQ_n} = \overrightarrow{OM_n}, OQ_n = OM_n = \sqrt{x_n(t)^2 + y_n(t)^2}, n \in \mathbb{Z}^+$, and $OA = \sqrt{p^2 + q^2}$. Consequently, consider the following extension with extra weights of coefficients r and s as follows: We therefore, consider the following scenario with extra weights of coefficients r and s as follows:

Theorem 1 *Given a non-zero closed curve C: $[x(t), y(t)]$, and a non-zero fixed point $A = (p, q) \notin C$. Let Q be a moving point on C. For $r, s > 0$, and $\overrightarrow{OM_1} = \frac{s \cdot OQ}{r \cdot OA + s \cdot OQ}\overrightarrow{OA} + \frac{r \cdot OA}{r \cdot OA + s \cdot OQ}\overrightarrow{OQ}$, if we write the Eq. (2) as*

$$\overrightarrow{OM_{n+1}} = \begin{bmatrix} x_{n+1}(t) \\ y_{n+1}(t) \end{bmatrix} = \frac{s \cdot OQ_n}{r \cdot OA + s \cdot OQ_n}\overrightarrow{OA} + \frac{r \cdot OA}{r \cdot OA + s \cdot OQ_n}\overrightarrow{OQ_n}. \tag{3}$$

Then $\overrightarrow{OM_{n+1}}$ converges for some $t \in [0, 2\pi]$ when $n \to \infty$ if and only if either $OQ_n = \sqrt{x_n^2(t) + y_n^2(t) + z_n^2(t)} \to 0$ or $\overrightarrow{OM_n(t)} \to \overrightarrow{OA}$ for some $t \in [0, 2\pi]$ when $n \to \infty$.

Proof: First, if $\overrightarrow{OM_{n+1}}$ converges for some $t \in [0, 2\pi]$ when $n \to \infty$, then $\overrightarrow{M_nM_{n+1}} = \overrightarrow{OM_{n+1}} - \overrightarrow{OM_n} \to 0$ for some $t \in [0, 2\pi]$ when $n \to \infty$. Moreover, since

$$\begin{aligned}
\overrightarrow{M_nM_{n+1}} &= \overrightarrow{OM_{n+1}} - \overrightarrow{OM_n} \\
&= \frac{s \cdot OQ_n}{r \cdot OA + s \cdot OQ_n}\overrightarrow{OA} + \frac{r \cdot OA}{r \cdot OA + s \cdot OQ_n}\overrightarrow{OM_n} - \overrightarrow{OM_n} \\
&= \frac{s \cdot OQ_n}{r \cdot OA + s \cdot OQ_n}\overrightarrow{OA} + \overrightarrow{OM_n}\left(\frac{r \cdot OA}{r \cdot OA + s \cdot OQ_n} - 1\right) \\
&= \frac{s \cdot OQ_n}{r \cdot OA + s \cdot OQ_n}\overrightarrow{OA} + \overrightarrow{OM_n}\left(\frac{-s \cdot OQ_n}{r \cdot OA + s \cdot OQ_n}\right). \\
&= \frac{s \cdot OQ_n}{r \cdot OA + s \cdot OQ_n}\left(\overrightarrow{OA} - \overrightarrow{OM_n}\right). \qquad (4)
\end{aligned}$$

Hence, $\overrightarrow{OM_{n+1}} = \begin{bmatrix} x_{n+1}(t) \\ y_{n+1}(t) \end{bmatrix}$ converges for some $t \in [0, 2\pi]$ if either $OQ_n = \sqrt{x_n^2(t) + y_n^2(t) + z_n^2(t)} \to 0$ or $\overrightarrow{OM_n(t)} \to \overrightarrow{OA}$ for some $t \in [0, 2\pi]$ when $n \to \infty$. The other direction is clear. ■

We describe a special convex combination of vectors in the vector space $\mathbb{R}^n$ below.

Definition 2 *Given a finite number of vectors $v_1, v_2, \ldots\ v_n$ in $\mathbb{R}^n$, a conical combination of these vectors is vector of the form*

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_n v_n,$$

where $\alpha_i > 0, i = 1, 2, \ldots n$. A set of conical combination of vectors is called a ***convex combination*** *[2] if in addition the coefficient satisfying the following condition*

$$\sum_{i=1}^{n} \alpha_i = 1.$$

In this paper, we shall discuss a special weighted convex combination of vectors that involve a recursive sequence. For example, if

$$\begin{aligned}
\overrightarrow{OM_{n+1}(t)} &= \begin{bmatrix} x_{n+1}(t) \\ y_{n+1}(t) \end{bmatrix} = \frac{\alpha_1}{\alpha_1 + \alpha_2 + \alpha_3}v_1 + \frac{\alpha_2}{\alpha_1 + \alpha_2 + \alpha_3}v_2 \\
&\quad + \frac{\alpha_3}{\alpha_1 + \alpha_2 + \alpha_3}\overrightarrow{OM_n(t)}, \qquad (5)
\end{aligned}$$

then α_1, α_2 and α_3 are positive real numbers. Using the scaling techniques, without loss of generality, we assume α_1, α_2 and α_3 are real numbers in $(0, 1)$. We shall see in later proofs that the coefficient α_3 is irrelevant to the convergence of $\lim_{n\to\infty} \overrightarrow{OM_{n+1}(t)}$.

2 2D iterations

2.1 One curve and one fixed vector

For the rest of the paper, we assume the fixed point A is not on the original curve C. In view of the Theorem (1), we further extend the knowledge of uniform convergence of sequences of functions, which students learn in Advanced Calculus. We begin with the domain $D = [0, 2\pi]$, and $\{M_n : D \to \mathbb{R}^2\}$ being a sequence of functions, and note that since the metric space $\mathbb{R}^2$ is complete, which means that every uniformly Cauchy sequence M_n is convergent. We consider the following:

Definition 3 *Suppose $D = [0, 2\pi]$, and $\{M_n : D \to \mathbb{R}^2\}$ is a sequence of functions. If we write $M_n(t) = [x_n(t), y_n(t)]$, with $t \in [0, 2\pi]$, $\{M_n(t))$ is said to* ***converge uniformly*** *to $M^*(t) = [p(t), q(t)]$ if $\forall \epsilon > 0$, $\exists$ a positive integer $N = N(\epsilon)$ (i.e. N depends only on ϵ in this case) such that the Euclidean distance between two points, $M_n(t)$ and $M^*(t)$, $\|M_n(t) - M^*(t)\|$ or $\|M_n(t)\, M^*(t)\|$, is arbitrarily small:*

$$\|M_n(t) - M^*(t)\| = \|M_n(t)\, M^*(t)\| = \sqrt{(x_n(t) - p(t))^2 + (y_n(t) - q(t))^2} < \epsilon.$$

Similarly, the sequence $\{M_n(t))$ is said to ***converge uniformly*** *to a point $A = (p, q)$ if $\forall \epsilon > 0$, $\exists$ a positive integer $N = N(\epsilon)$ such that $\|M_n(t)\, A\|$ is arbitrarily small. In other words,*

$$\|M_n(t)\, A\| = \sqrt{(x_n(t) - p)^2 + (y_n(t) - q)^2} < \epsilon$$

for all $n \geq N$ and all $t \in [0, 2\pi]$. Intuitively, there exists a positive integer N, such that the parametric curves $M_n(t)$ will shrink to the point A for all $n \geq N$ and all $t \in [0, 2\pi]$.

Definition 4 *Suppose $D = [0, 2\pi]$, and $\{M_n : D \to \mathbb{R}^2\}$ is a sequence of functions. If we write $M_n(t) = [x_n(t), y_n(t)]$, with $t \in [0, 2\pi]$, $\{M_n(t))$ is said to be uniformly Cauchy if for every $\varepsilon > 0$, there exists a positive integer N such that the inequality*

$$\|M_n(t)\, M_m(t)\| < \varepsilon$$

holds whenever $m \geq N$, $n \geq N$, and for all $t \in D$. We take it for granted in this paper that the sequence $\{M_n : D \to \mathbb{R}^2\}$ converges uniformly to another M on D if and only if, the sequence $\{M_n)$ is uniformly Cauchy.

Remarks:

1. We remark that definitions in (3) and in (4) can be extended to $\mathbb{R}^n$.

2. We remind readers to distinguish the difference between uniform convergence versus point-wise convergence.

3. Recall our original bisection problem (1) is such that $\frac{M_1A}{M_1Q_0} = \frac{AB}{OB} = \frac{AB}{M_1B} = \frac{OA}{OQ_0} = k_1(t)$, where the convergence in the case of (2) is a homothety (see [3]). We may denote the following:
$$\frac{M_nA}{M_nM_{n-1}} = k_n(t) \left(= \frac{OA}{OM_{n-1}} \right), \tag{6}$$
where $n = 1, 2, ...$, and $M_0 = Q$, which is a point on the given curve C.

4. On one hand, we usually prove how a sequence of parametric curves $\{M_n(t)\}_{n=1}^{\infty}$ converge uniformly directly in this paper. On the other hand, we note that $\{M_n(t)\}$ is a sequence from $D=[0,2\pi]$ to $\mathbb{R}^2$, and since $\mathbb{R}^2$ is a complete metric space, if one can show that $\{M_n(t)\}$ is a uniformly Cauchy sequence, then $\{M_n(t)\}$ is uniformly convergent. Instead of proving that $\{M_n(t)\}$ is a uniformly Cauchy sequence theoretically in this paper, with the help of a CAS, we often demonstrate that the graph of square distance function $f_n(t)=\sup\left(\|M_n(t)-M_{n-1}(t)\|\right)^2$ or $g_n(t)=\sup\left(\|M_n(t)-A\|\right)^2$, for all $t\in D=[0,2\pi]$, is decreasing to 0 uniformly, and use such observation to conjecture that $\{M_n(t)\}_{n=1}^{\infty}$ converges uniformly.

The next observation is natural:

Theorem 5 *Let C be a given simple closed curve $[x_0(t),y_0(t)]$, $A=(p_1,q_1)\notin C$. For $r,s\in(0,1)$ and $r\neq s$, we let*

$$\overrightarrow{OM_1}=\begin{bmatrix}x_1(t)\\ y_1(t)\end{bmatrix}=\frac{s\cdot OQ}{r\cdot OA+s\cdot OQ}\overrightarrow{OA}+\frac{r\cdot OA}{r\cdot OA+s\cdot OQ}\overrightarrow{OQ},$$

where Q is a moving point on C. Now for $n\in\mathbb{Z}^+$, we consider

$$\overrightarrow{OM_{n+1}}=\begin{bmatrix}x_{n+1}(t)\\ y_{n+1}(t)\end{bmatrix}=\frac{s\cdot OQ_n}{r\cdot OA+s\cdot OQ_n}\overrightarrow{OA}+\frac{r\cdot OA}{r\cdot OA+s\cdot OQ_n}\overrightarrow{OQ_n}, \tag{7}$$

where Q_n is a moving point on $(x_n(t),y_n(t))$, and $\overrightarrow{OQ_n(t)}=\overrightarrow{OM_n(t)}$. Then $\overrightarrow{OM_n(t)}\to\overrightarrow{OA}$ uniformly as $n\to\infty$ for all $t\in[0,2\pi]$, $\overrightarrow{M_{n-1}(t)M_n(t)}$ converges uniformly to 0 for all $t\in[0,2\pi]$. Consequently, $\{M_n(t)\}_{n=1}^{\infty}$ converges to A uniformly.

Proof: First, if $r=s$ and $r,s\in(0,1)$, we refer to Theorem (1) for discussion. Now, for $r,s\in(0,1)$ and $r\neq s$,

$$\overrightarrow{OM_1}=\begin{bmatrix}x_1(t)\\ y_1(t)\end{bmatrix}=\frac{s\cdot OQ}{r\cdot OA+s\cdot OQ}\overrightarrow{OA}+\frac{r\cdot OA}{r\cdot OA+s\cdot OQ}\overrightarrow{OQ},$$

we first observe that $M_n=Q_n=(x_n(t),y_n(t))$ for $n\geq 1$, and

$$\begin{aligned}\overrightarrow{OM_2} &= \begin{bmatrix}x_2(t)\\ y_2(t)\end{bmatrix}=\frac{s\cdot OQ_1}{r\cdot OA+s\cdot OQ_1}\overrightarrow{OA}+\frac{r\cdot OA}{r\cdot OA+s\cdot OQ_1}\begin{bmatrix}x_1(t)\\ y_1(t)\end{bmatrix}\\ &= \frac{s\cdot OQ_1}{r\cdot OA+s\cdot OQ_1}\overrightarrow{OA}+\frac{r\cdot OA}{r\cdot OA+s\cdot OQ_1}\left(\frac{s\cdot OQ}{r\cdot OA+s\cdot OQ}\overrightarrow{OA}+\frac{r\cdot OA}{r\cdot OA+s\cdot OQ}\overrightarrow{OQ}\right)\\ &= \overrightarrow{OA}\left(\frac{(rs)\left[(OA)(OQ)+(OA)\,OQ_1\right]+s^2(OQ)(OQ_1)}{(r\cdot OA+s\cdot OQ_1)(r\cdot OA+s\cdot OQ)}\right)\\ &\quad+\overrightarrow{OQ}\left(\frac{r^2\cdot(OA)^2}{(r\cdot OA+s\cdot OQ_1)(r\cdot OA+s\cdot OQ)}\right).\end{aligned}$$

By induction, we see

$$\begin{aligned}\overrightarrow{OM_{n+1}} &= \overrightarrow{OA}\left(\frac{(r\cdot OA+s\cdot OQ_n)\cdots(r\cdot OA+s\cdot OQ_1)(r\cdot OA+s\cdot OQ)-r^n\cdot(OA)^n}{(r\cdot OA+s\cdot OQ_n)\cdots(r\cdot OA+s\cdot OQ_1)(r\cdot OA+s\cdot OQ)}\right)\\ &\quad+\overrightarrow{OQ_n}\left(\frac{r^n\cdot(OA)^n}{(r\cdot OA+s\cdot OQ_n)\cdots(r\cdot OA+s\cdot OQ_1)(r\cdot OA+s\cdot OQ)}\right)\end{aligned}\tag{8}$$

Since $0 < r < 1$,

$$\frac{r^n \cdot (OA)^n}{(r \cdot OA + s \cdot OQ_n) \cdots (r \cdot OA + s \cdot OQ_1)(r \cdot OA + s \cdot OQ)} \to 0.$$

Furthermore, since $\overrightarrow{OM_{n+1}} = a\overrightarrow{OA} + b\overrightarrow{OQ_n}$, where a and b are coefficients of $\overrightarrow{OA}$ and $\overrightarrow{OQ_n}$ respectively as seen in Eq. (8) with $a, b \in (0,1)$ and $a+b = 1$, this implies that $\overrightarrow{OM_{n+1}(t)} \to \overrightarrow{OA}$ as $n \to \infty$ for all $t \in [0, 2\pi]$. Since three points, $M_{n-1}(t)$, $M_n(t)$ and A are collinear, and $M_n(t)$ is in the interior of $M_{n-1}(t)$ and A, we see $\overrightarrow{M_{n-1}(t)M_n(t)}$ converges uniformly to 0 for all $t \in [0, 2\pi]$, which can be shown that $\overrightarrow{M_n(t)}$ is uniformly Cauchy, and hence $\{M_n(t)\}_{n=1}^{\infty}$ converges to A uniformly. ■

We remark that the uniform convergence of $\{M_n(t)\}_{n=1}^{\infty}$ to the point A does not depend on the curve C.

Example 6 *We consider the curve C of $[a\cos(t), b\sin t]$, $A = (p_1, q_1) \notin C$, For the convex combination of r and s, we let*

$$\begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} = \frac{s \cdot OQ}{r \cdot OA + s \cdot OQ} \begin{bmatrix} p_1 \\ q_1 \end{bmatrix} + \frac{r \cdot OA}{r \cdot OA + s \cdot OQ} \begin{bmatrix} x_0(t) \\ y_0(t) \end{bmatrix},$$

and

$$\overrightarrow{OM_{n+1}} = \begin{bmatrix} x_{n+1}(t) \\ y_{n+1}(t) \end{bmatrix} = \frac{s \cdot OQ_n}{r \cdot OA + s \cdot OQ_n}\overrightarrow{OA} + \frac{r \cdot OA}{r \cdot OA + s \cdot OQ_n}\overrightarrow{OQ_n}.$$

If we choose $a = 5, b = 4$, and convex combination for $r = \frac{1}{3}, s = \frac{2}{3}$, $A = (3,2)$, then $\{M_n(t)\}_{n=1}^{\infty}$ converges to A uniformly. (See Figure 2)

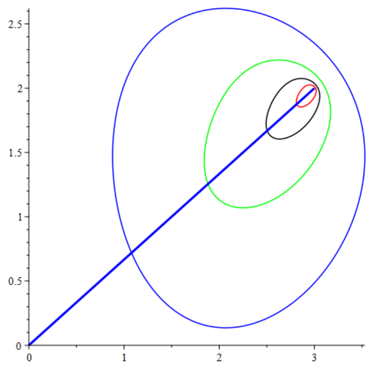


Figure 2. Uniform converges to a point.

Exercises: (1) If we use $r = s$ in Example (6), then we leave it to the readers to verify that $g_n(t) = (\|M_n(t) - A\|)^2$ does not converge uniformly to 0. In fact, the maximum value of $g_n(t)$ is the distance $(OA)^2$ at some $t \in (0, 2\pi)$. (2) If we replace C by $[a\sin t, b\sin t\cos t]$, $a = 5, b = 4$, $r = \frac{1}{3}, s = \frac{2}{3}$, $A = (3,2)$ in Example (6), then we may conjecture that $\{M_n(t)\}_{n=1}^{\infty}$ does not converge to A uniformly by observing the graph of $f_n(t) = \|M_n(t) - M_{n-1}(t)\|$ does not converge uniformly to 0.

2.2 One curve and two fixed vectors

We consider convex combinations of three vectors below: Let C be a given closed curve $[x_0(t), y_0(t)]$, $A = (p_1, q_1)$ and $B = (p_2, q_2)$ be two distinct points not lying on C. If Q is a moving point on C, and r_1, r_2, and r_3 are real numbers in $(0, 1)$. For $n \in \mathbb{Z}^+ \cup \{0\}$, we consider

$$\begin{aligned}\overrightarrow{OM_{n+1}} &= \begin{bmatrix} x_{n+1}(t) \\ y_{n+1}(t) \end{bmatrix} = \frac{r_1 \cdot OQ_n}{r_1 OQ_n + r_2 OA + r_3 OB}\overrightarrow{OA} + \frac{r_2 \cdot OA}{r_1 OQ_n + r_2 OA + r_3 OB}\overrightarrow{OB} \\ &\quad + \frac{r_3 \cdot OB}{r_1 OQ_n + r_2 OA + r_3 OB}\overrightarrow{OM_n},\end{aligned}$$

where $M_0(t) = Q(t) \in C$, and $M_n(t) = Q_n(t)$ is a moving point on $(x_n(t), y_n(t))$. We are interested in $\lim_{n\to\infty} \overrightarrow{OM_{n+1}}$.

2.3 Generating sequence of shrinking curves due to convex combinations

Since the plot of the sequence $\overrightarrow{OM_{n+1}}$ in (9), where r_1, r_2, and r_3 are distinct real numbers in $(0, 1)$, is a convex combinations of vectors $\overrightarrow{OA}, \overrightarrow{OB}$ and $\overrightarrow{OM_n}$, the plot of $[x_{n+1}(t), y_{n+1}(t)]$ is generated by the following steps:

1. Connect three points of $M_n = (x_n(t), y_n(t))$, A and B to form the triangle $\triangle M_n AB$.

2. We view the point M_n as the convex combination of three points A, B and M_{n-1}, for $n \in \mathbb{Z}^+$, where $M_0 = Q$, which is a point on the curve C. Since r_1, r_2, and $r_3 \in (0, 1)$, the point $M_n(t)$ belongs to the interior of the triangle $\triangle M_{n-1}AB$ for each $t \in [0, 2\pi]$, and $n \in \mathbb{Z}^+$, see [2].

3. We shall see later in the proof of the Theorem (8) that the coefficient r_3 will not affect the final plot of $\overrightarrow{OM_n}$ when $n \to \infty$.

4. The convergence of $\overrightarrow{OM_n}$ will only depend on $\overrightarrow{OA}$ and $\overrightarrow{OB}$, and will not depend on the curve C.

Example 7 *We use closed curve C to be $[a \sin u, b \sin u \cos u]$, $a = 5, b = 4$, $A = (3, 4)$, $B = (2, 5)$, $r_1 = \frac{1}{2}, r_2 = \frac{1}{3}$, and $r_3 = \frac{1}{6}$ for demonstrating how $[x_2(t), y_2(t)]$ is generated from $[x_1(t), y_1(t)]$. The graphs of $[x_1(t), y_1(t)]$ and $[x_2(t), y_2(t)]$ can be seen in black and purple respectively in Figure 3 (d) respectively.*

- Figure 3(a) shows when $t = 0$,the plot of $[x_2(t), y_2(t)]$ has not been generated yet.

- Figure 3(b) shows when $t \in [0, 0.9106]$, the plot of $[x_2(t), y_2(t)]$ is being generated in this interval and will be in the interior of $\triangle M_1 AB$ for each corresponding t.

- Figure 3(c) shows when $t \in [0, 3.1871]$, the plot of $[x_2(t), y_2(t)]$ is being generated in this interval and will be in the interior of $\triangle M_1 AB$ for each corresponding t, and finally, Figure

4(d) shows when $t \in [0, 2\pi]$, the plot of $[x_2(t), y_2(t)]$ is smaller than that of $[x_1(t), y_1(t)]$.

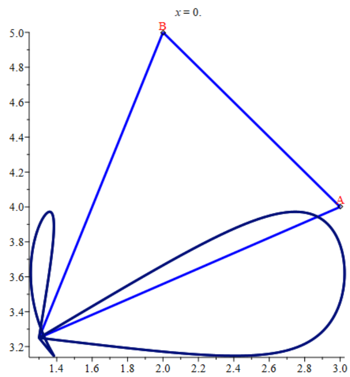


Figure 3(a), $t = 0$.

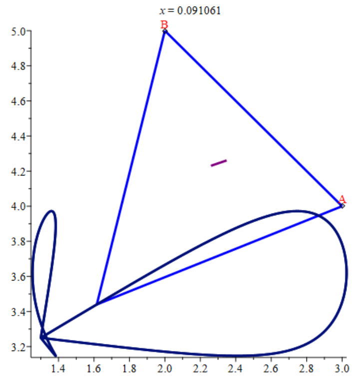


Figure 3(b), $t \in [0, 0.9106]$.

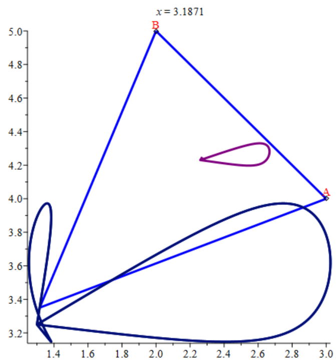


Figure 3(c), $t \in [0, 3.1871]$.

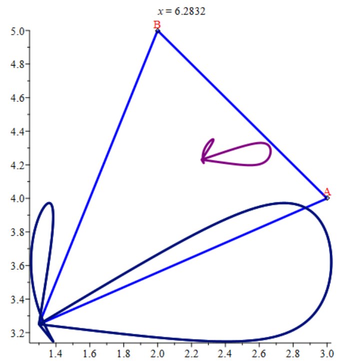


Figure 3(d), $t \in [0, 2\pi]$.

Theorem 8 *Let C be a given closed curve $[x_0(t), y_0(t)]$, $A = (p_1, q_1)$ and $B = (p_2, q_2)$ be two non-zero distinct points not lying on C. If Q is a moving point on C, and r_1, r_2, and r_3 are positive real numbers in $(0,1)$, we let*

$$\begin{aligned}\overrightarrow{OM_1} &= \begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} = \frac{r_1 \cdot OQ}{r_1 OQ + r_2 OA + r_3 OB}\overrightarrow{OA} + \frac{r_2 \cdot OA}{r_1 OQ + r_2 OA + r_3 OB}\overrightarrow{OB} \\ &\quad + \frac{r_3 \cdot OB}{r_1 OQ + r_2 OA + r_3 OB}\overrightarrow{OQ}.\end{aligned}$$

We further consider

$$\begin{aligned}\overrightarrow{OM_{n+1}} &= \begin{bmatrix} x_{n+1}(t) \\ y_{n+1}(t) \end{bmatrix} = \frac{r_1 \cdot OQ_n}{r_1 OQ_n + r_2 OA + r_3 OB}\overrightarrow{OA} + \frac{r_2 \cdot OA}{r_1 OQ_n + r_2 OA + r_3 OB}\overrightarrow{OB} \\ &\quad + \frac{r_3 \cdot OB}{r_1 OQ_n + r_2 OA + r_3 OB}\begin{bmatrix} x_n(t) \\ y_n(t) \end{bmatrix}, \qquad (9)\end{aligned}$$

where Q_n is a moving point on $(x_n(t), y_n(t))$. Then $\{M_n(t)\}_{n=1}^{\infty}$ converges uniformly to a point D, ***which lies on the line segment*** *$\overline{AB}$. Consequently, $\overrightarrow{M_{n-1}(t)M_n(t)}$ converges uniformly to 0 for all $t \in [0, 2\pi]$. We remark that the coefficient $r_3 \in (0,1)$ will not affect the location of the convergence $\{M_n(t)\}_{n=1}^{\infty}$.*

Proof: First, we observe

$$\begin{aligned}\overrightarrow{OM_2} &= \begin{bmatrix} x_2(t) \\ y_2(t) \end{bmatrix} = \frac{r_1 \cdot OQ_1}{r_1OQ_1 + r_2OA + r_3OB}\overrightarrow{OA} + \frac{r_2 \cdot OA}{r_1OQ_1 + r_2OA + r_3OB}\overrightarrow{OB} \\ &\quad + \frac{r_3 \cdot OB}{r_1OQ_1 + r_2OA + r_3OB}\left(\begin{array}{c} \frac{r_1 \cdot OQ}{r_1OQ + r_2OA + r_3OB}\overrightarrow{OA} + \frac{r_2 \cdot OA}{r_1OQ + r_2OA + r_3OB}\overrightarrow{OB} \\ + \frac{r_3 \cdot OB}{r_1OQ + r_2OA + r_3OB}\overrightarrow{OQ} \end{array} \right) \\ &= \overrightarrow{OA}\left(\left\|\overrightarrow{OA}\right\|\right) + \overrightarrow{OB}\left(\left\|\overrightarrow{OB}\right\|\right) + \overrightarrow{OQ}\left(\frac{r_3^2 (OB)^2}{(r_1OQ_1 + r_2OA + r_3OB)(r_1OQ + r_2OA + r_3OB)}\right),\end{aligned}$$

It follows from induction that

$$\begin{aligned}\overrightarrow{OM_{n+1}} &= \overrightarrow{OA}\left(\left\|\overrightarrow{OA}\right\|\right) + \overrightarrow{OB}\left(\left\|\overrightarrow{OB}\right\|\right) \\ &\quad + \overrightarrow{OQ}\left(\frac{r_3^n (OB)^n}{(r_1OQ_n + r_2OA + r_3OB)\cdots(r_1OQ_1 + r_2OA + r_3OB)(r_1OQ + r_2OA + r_3OB)}\right).\end{aligned}$$

Since $0 < r_3 < 1$, we see $r_3^n (OB)^n \to 0$, and

$$\overrightarrow{OM_{n+1}} \to m\overrightarrow{OA} + (1-m)\overrightarrow{OB},$$

when $n \to \infty$, where $m = \left\|\overrightarrow{OA}\right\|$, and $1 - m = \left\|\overrightarrow{OB}\right\|$. Let $D = m\overrightarrow{OA} + (1-m)\overrightarrow{OB}$, then $D \in \overline{AB}$, and $\overrightarrow{OM_{n+1}}$ converges uniformly to $\overrightarrow{OD}$. Hence $\overrightarrow{OM_{n+1}}$ converges uniformly to $\overrightarrow{OD}$, where $D \in \overline{AB}$. In view of the observations from section (2.3), we see $\{M_n(t)\}_{n=1}^{\infty}$ converges uniformly to the point D, which lies on the line segment $\overline{AB}$. Moreover, it is clear that $\overrightarrow{M_{n-1}(t)M_n(t)} = \overrightarrow{OM_n} - \overrightarrow{OM_{n-1}}$ converges uniformly to 0 for all $t \in [0, 2\pi]$, ■

Computationally, we assume $\begin{bmatrix} x_{n+1}(t) \\ y_{n+1}(t) \end{bmatrix} \to F = \begin{bmatrix} p \\ q \end{bmatrix}$, then the norm of the vector, $\left\|\begin{bmatrix} x_{n+1}(t) \\ y_{n+1}(t) \end{bmatrix}\right\|$, converges to $\|F\| = \sqrt{p^2+q^2}$, and we have

$$\left(1 - \frac{r_3OB}{r_1\|F\| + r_2OA + r_3OB}\right)\begin{bmatrix} p \\ q \end{bmatrix} = \frac{r_1\|F\|}{r_2OA + r_3OB + r_1\|F\|}\overrightarrow{OA} + \frac{r_2OA}{r_2OA + r_3OB + r_1\|F\|}\overrightarrow{OB}$$

$$\begin{aligned}\begin{bmatrix} p \\ q \end{bmatrix} &= \left(\frac{1}{\left(\frac{r_1\|F\| + r_2OA}{r_1\|F\| + r_2OA + r_3OB}\right)}\right)\left(\frac{r_1\|F\|}{r_2OA + r_3OB + r_1\|F\|}\overrightarrow{OA} + \frac{r_2OA}{r_2OA + r_3OB + r_1\|F\|}\overrightarrow{OB}\right) \\ &= \left(\frac{r_1\|F\|}{r_1\|F\| + r_2OA}\right)\overrightarrow{OA} + \left(\frac{r_2OA}{r_1\|F\| + r_2OA}\right)\overrightarrow{OB} \\ &= m\overrightarrow{OA} + (1-m)\overrightarrow{OB}, \qquad (10)\end{aligned}$$

where $m = \frac{r_1\|F\|}{r_1\|F\|+r_2 OA}$. To find the point F, it amounts to solve two equations in (10) for two variables p and q in terms of t; however, due to too many parameters that are involved, we are unable to express the solutions p and q in explicit form. Instead, we do the followings:

1. If r_1, r_2, and r_3 are real numbers in $(0,1)$, we substitute the solutions p and q obtained (10) into the line equation $\overleftrightarrow{AB}$, we get the following equation from Maple after setting **the length of computations** to be 20,000 lines:

$$\frac{(q-q_2)\,p_1+(q_1-q)\,p_2-p\,(q_1-q_2)}{p_1-p_2}=0$$

$$\begin{aligned} \Longrightarrow \quad & \frac{qp_1-pq_1+pq_2-qp_2-p_1q_2+p_2q_1}{p_1-p_2}=0, \\ \Rightarrow \quad & \frac{q(p_1-p_2)-p(q_1-q_2)-p_1q_2+p_2q_1}{p_1-p_2}=0. \end{aligned} \tag{11}$$

2. Assume $p_1 \neq p_2$ we deduce the numerator of (11) be to the following:

$$\begin{aligned} q(p_1-p_2)-p(q_1-q_2)-p_1q_2+p_2q_1 &= 0, \\ \frac{q(p_1-p_2)-p(q_1-q_2)-p_1q_2+p_2q_1}{p_1-p_2} &= 0, \\ q-p\left(\frac{q_1-q_2}{p_1-p_2}\right)-\frac{p_1q_2-p_2q_1}{p_1-p_2} &= 0. \end{aligned}$$

On the one hand, we see $F=(p,q)$ lies on the line of

$$y=\left(\frac{q_1-q_2}{p_1-p_2}\right)x+\frac{p_1q_2-p_2q_1}{p_1-p_2}. \tag{12}$$

On the other hand, we note that the line $\overleftrightarrow{AB}$ is with the slope $\frac{q_1-q_2}{p_1-p_2}$ and passes through the point (p_1,q_1) :

$$\begin{aligned} y-q_1 &= \left(\frac{q_1-q_2}{p_1-p_2}\right)(x-p_1) \\ y &= q_1+\left(\frac{q_1-q_2}{p_1-p_2}\right)(x-p_1) \\ &= \left(\frac{q_1-q_2}{p_1-p_2}\right)x+\frac{p_1q_2-q_1p_2}{p_1-p_2}. \end{aligned} \tag{13}$$

We see (12) coincides with (13) and hence F lie on line segment $\overline{AB}$. We remark that when solving p and q symbolically if r_1, r_2 and r_3 are also considered to be variables, it is not possible to express using p and q due to too many unknowns when using [1], but numerical computations do show that the point (p,q) lie on the line segment AB. We use the following Example for demonstration.

Example 9 *We consider the closed curve* C_1 *with the parametric equation,* $[x_0(t), y_0(t)] = [\cos u(a - \cos(bu)) + 1, \sin u(a - \cos bu))]$, $A = (p_1, q_1)$, $B = (p_2, q_2)$, *and* Q *is a moving point on* C_1. *We let* $r_1, r_2,$ *and* r_3 *be three distinct real numbers in* $(0, 1)$, *and*

$$\overrightarrow{OM_{n+1}} = \begin{bmatrix} x_{n+1}(t) \\ y_{n+1}(t) \end{bmatrix} = \frac{r_1 \cdot OQ_n}{r_1 OQ_n + r_2 OA + r_3 OB} \overrightarrow{OA} + \frac{r_2 \cdot OA}{r_1 OQ_n + r_2 OA + r_3 OB} \overrightarrow{OB} + \frac{r_3 \cdot OB}{r_1 OQ_n + r_2 OA + r_3 OB} \begin{bmatrix} x_n(t) \\ y_n(t) \end{bmatrix}.$$

If we pick $a = 5, b = 4$, $p_1 = 3, q_1 = 4, p_2 = 2, q_2 = 5$, and $r_1 = \frac{1}{2}, r_2 = \frac{1}{3}$, and $r_3 = \frac{1}{6}$. Then we see

$$\lim_{n \to \infty} \{M_n(t)\}_{n=1}^{\infty} = (2.60516252, 4.39483748),$$

see Figure 4(a) below for the convergence. In view of (10), we note that the convergence does not depend on the value of r_3.We also remark that convergence to the point $(2.60516252, 4.39483748)$ is irrespective to the curve C_1 we pick. For example, if we replace C_2 by $[a \sin u, b \sin u \cos u]$, and use the same a, b, point A, and point B, we shall get the same convergence for $\lim_{n\to\infty} \{M_n(t)\}_{n=1}^{\infty} = (2.60516252, 4.39483748)$, (see Figure 4(b)). Similarly is true if we replace C_3 by $\left[4a \cos u (\sin u)^2 \cos u, 4a$ co
see (Figure 4(c)).

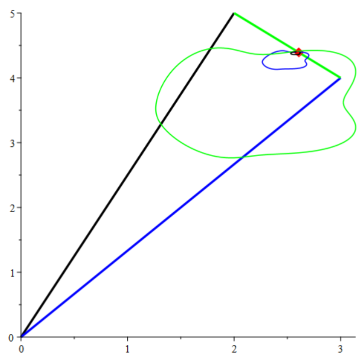


Figure 4(a). Convergence for C_1.

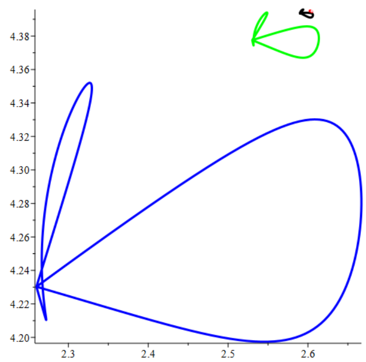


Figure 4(b). Convergence for C_2.

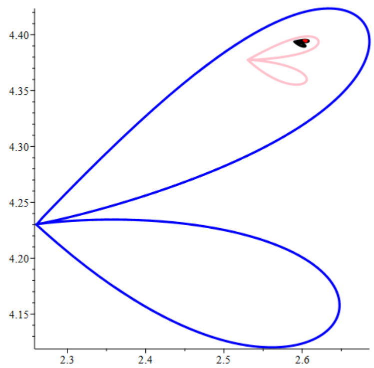


Figure 4(c). Convergence for C_3.

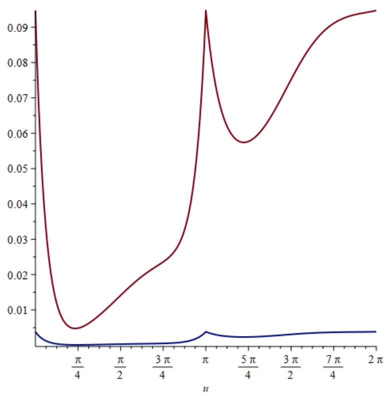
2.4 Uniform convergence using geometric constructions

In view of the Theorem (8) and observation from section (2.3), C is a non-zero closed curve, A and B are two non-zero distinct fixed points, not lying on C, and $M_n(t)$ is in the interior of the triangle of $\triangle M_{n-1}(t) AB$ for each $t \in [0, 2\pi]$. We see the distance between $M_n(t) = [x_n(t), y_n(t)]$ and $M_{n-1}(t) = [x_{n-1}(t), y_{n-1}(t)]$ is decreasing and converges to 0 when $n \to \infty$, for all $t \in [0, 2\pi]$. In other words, the square distance function

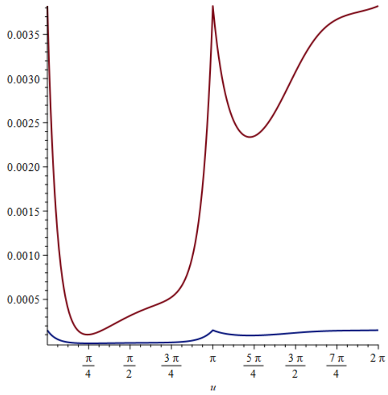
$$f_n(t) = (x_n(t) - x_{n-1}(t))^2 + (y_n(t) - y_{n-1}(t))^2$$

converges to 0 uniformly. Consequently, we see $\{M_n(t)\}_{n=1}^{\infty}$ converges to a point lying on the line segment $\overline{AB}$. In other words, if the graphs of $f_n(t)$ does not converges to 0 uniformly, then $\overrightarrow{OM_{n+1}}$ does not converge uniformly.

Suppose we adopt the Example in the section (2.3), we depict the pair functions $\{f_3(t), f_4(t)\}$ and $\{f_4(t), f_5(t)\}$ in the following Figures 5(a) and 5(b) with red and blue colors respectively:



Figures 5(a). Plots of $\{f_3(t), f_4(t)\}$.



Figures 5(b). Plots of $\{f_4(t), f_5(t)\}$

In view of the plot of $f_5(t)$ (the blue in Figure 5(b)), we can see that if we pick $\epsilon = 0.0005$, for $n \geq 5$, $f_n(t) \to 0$ uniformly for all $t \in [0, 2\pi]$. In view of the Example (9), the speed of the uniform convergence of $\lim_{n\to\infty} \{M_n(t)\}_{n=1}^{\infty} = (2.60516252, 4.39483748)$ is rather fast.

2.5 Iterations on one curve, and two vectors on two respective curves

Now, we consider the plots of convex combinations of three vectors, one vector is iterated one curve, and two vectors are on two respective curves.

Theorem 10 *Let C be a given non-zero closed curve $[x_0(t), y_0(t)]$, D and E be two additional distinct closed curves of $[d_1(t), d_2(t)]$ and $[e_1(t), e_2(t)]$ respectively. Furthermore, we let Q be a moving point on C. If r_1, r_2, and r_3 are real numbers in $(0, 1)$, we let*

$$\begin{aligned} OQ &= \sqrt{x_0(t)^2 + y_0(t)^2}, \\ OE &= \sqrt{e_1(t)^2 + e_2(t)^2}, \\ OD &= \sqrt{d_1(t)^2 + d_2(t)^2}, \end{aligned}$$

and

$$\begin{aligned} \overrightarrow{OM_1} &= \begin{bmatrix} x_1(t) \\ y_1(t) \end{bmatrix} = \frac{r_1 \cdot OQ}{r_1 OQ + r_2 OE + r_3 OD}\overrightarrow{OE} + \frac{r_2 \cdot OE}{r_1 OQ + r_2 OE + r_3 OD}\overrightarrow{OD} \\ &\quad + \frac{r_3 \cdot OD}{r_1 OQ + r_2 OE + r_3 OD}\overrightarrow{OQ}. \end{aligned}$$

In addition, for $n \in \mathbb{Z}^+$, we consider

$$\begin{aligned} \overrightarrow{OM_{n+1}} &= \begin{bmatrix} x_{n+1}(t) \\ y_{n+1}(t) \end{bmatrix} = \frac{r_1 \cdot OQ_n}{r_1 OQ_n + r_2 OE + r_3 OD}\overrightarrow{OE} + \frac{r_2 \cdot OE}{r_1 OQ_n + r_2 OE + r_3 OD}\overrightarrow{OD} \\ &\quad + \frac{r_3 \cdot OD}{r_1 OQ_n + r_2 OE + r_3 OD}\begin{bmatrix} x_n(t) \\ y_n(t) \end{bmatrix}, \end{aligned} \quad (14)$$

where Q_n is a moving point on $(x_n(t), y_n(t))$, for $n = 0, 1, ...$ If $\{M_n(t)\}_{n=1}^{\infty}$ converges, then $\{M_n(t)\}_{n=1}^{\infty}$ converges uniformly to the curve $F(t)$, which satisfies the solutions (15) and (16) for $t \in [0, 2\pi]$. Furthermore, the real solutions of the parametric curve is a subset of $\lim_{n\to\infty} \{M_n(t)\}_{n=1}^{\infty}$. We remark that the coefficient $r_3 \in (0, 1)$ will not affect where $\{M_n(t)\}_{n=1}^{\infty}$ will converge to.

Remark: Unlike the Theorem (8), where we make use of a decreasing sequence of closed convex sets, in this Theorem (10), we have two moving points on the curves D and E respectively. We may not have a decreasing sequence of closed convex sets, therefore, the assumption of $\{M_n(t)\}_{n=1}^{\infty}$ being convergent is needed. We shall explore ways of relaxing this condition in future paper.

Proof: We assume $\begin{bmatrix} x_{n+1}(t) \\ y_{n+1}(t) \end{bmatrix}$ converges to a real solution of $\begin{bmatrix} p(t) \\ q(t) \end{bmatrix}$, where $t \in [0, 2\pi]$. We see

$$\begin{bmatrix} p(t) \\ q(t) \end{bmatrix} \left(1 - \frac{r_3 \cdot OD}{r_1\sqrt{p(t)^2 + q(t)^2} + r_2 OE + r_3 OD} \right)$$
$$= \frac{r_1\sqrt{p(t)^2 + q(t)^2}}{r_1\sqrt{p(t)^2 + q(t)^2} + r_2 OE + r_3 OD}\overrightarrow{OE} + \frac{r_2 \cdot OE}{r_1\sqrt{p(t)^2 + q(t)^2} + r_2 OE + r_3 OD}\overrightarrow{OD}.$$

$$\begin{bmatrix} p(t) \\ q(t) \end{bmatrix} \frac{r_1\sqrt{p(t)^2 + q(t)^2} + r_2 OE}{r_1\sqrt{p(t)^2 + q(t)^2} + r_2 OE + r_3 OD}$$
$$= \left(\begin{array}{c} \frac{r_1\sqrt{p(t)^2+q(t)^2}}{r_1\sqrt{p(t)^2+q(t)^2}+r_2 OE+r_3 OD} \begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix} \\ +\frac{r_2 OE}{r_1\sqrt{p(t)^2+q(t)^2}+r_2 OE+r_3 OD} \begin{pmatrix} d_1(t) \\ d_2(t) \end{pmatrix} \end{array} \right)$$

It amounts to find the real solutions for $p(t)$ and $q(t)$ from the two equations (15) and (16) in terms of t, when r_1, r_2, r_3 are given.

$$\begin{bmatrix} p(t) \\ q(t) \end{bmatrix} = \frac{r_1\sqrt{p(t)^2 + q(t)^2}}{r_1\sqrt{p(t)^2 + q(t)^2} + r_2\sqrt{e_1(t)^2 + e_2(t)^2}} \begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix} \quad (15)$$
$$+\frac{r_2\sqrt{e_1(t)^2 + e_2(t)^2}}{r_1\sqrt{p(t)^2 + q(t)^2} + r_2\sqrt{e_1(t)^2 + e_2(t)^2}} \begin{pmatrix} d_1(t) \\ d_2(t) \end{pmatrix}. \blacksquare \quad (16)$$

In the next Example, we shall see how the graphs of the square distance functions can be used as a conjecture if the convergence of $\{M_n(t)\}_{n=1}^{\infty}$ is uniform. Secondly, we will see how the real solutions from solving for $p(t)$ and $q(t)$ computationally from the two equations (15) and (16) can serve as partial solution for the parametric curve $F(t) = \begin{bmatrix} p(t) \\ q(t) \end{bmatrix}$ under the uniform convergence of $\lim_{n\to\infty} \{M_n(t)\}_{n=1}^{\infty}$.

Example 11 *Let C be the given ellipse curve $[x_0(t), y_0(t)] = [a\cos t, b\sin t]$, D be the closed curve of*

$$[d_1(t), d_2(t)] = [(\sin 2t + 2)\cos t, (\sin 2t + 2)\sin t],$$

and E be the closed curve of

$$[(a - \cos(bt)\cos t + 1, (a - \cos(bt)\sin t].$$

Let Q be a moving point on C. We are interested in the plot of $\lim_{n\to\infty} M_n(t)$, see (14), when $n \to \infty$.

1. We consider $r_1 = \frac{1}{2}, r_2 = \frac{1}{3}, r_3 = \frac{1}{6}, a = 5, b = 3$. In addition, it is also worth noting that the square distance function

$$f_n(t) = (x_n(t) - x_{n-1}(t))^2 + (y_n(t) - y_{n-1}(t))^2$$

converges to 0 rather quickly in this case. We depict the pair functions $\{f_4(t), f_5(t)\}$ and $f_5(t)$ in the following Figures 6(a) and 6(b) respectively. Consequently, we may use these observations to conjecture that the convergence of $\{M_n(t)\}_{n=1}^{\infty}$ is uniform.

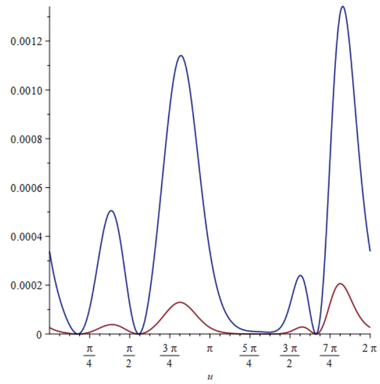


Figure 6(a). Plots of $\{f_4(t), f_5(t)\}$.

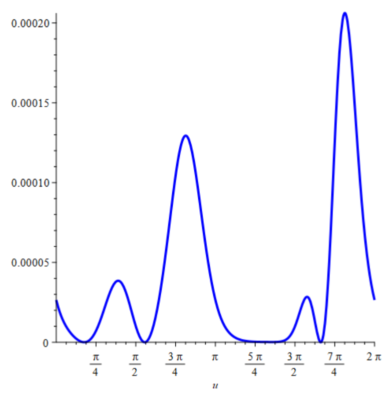


Figure 6(b). Plot of $f_5(t)$.

2. If we plot the real solutions of the branch 1, out of four branches when solving two equations (15) and (16), it coincides 'almost' exactly with that of $M_5(t) = \begin{bmatrix} x_5(t) \\ y_5(t) \end{bmatrix}$, see Figure 7 below, which we cannot tell them apart. See Supplementary Electronic Material

[S1].

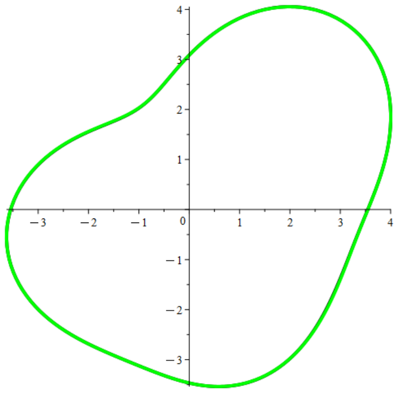


Figure 7. Graph of $M_5(t)$.

Exercise: We invite readers to explore that the plot of the curve $\lim_{n\to\infty} M_n(t)$, see (10), is invariant with the choice of curve $C = [x_0(t), y_0(t)]$.

3 Uniform Convergence of an Iterated Sequence in R^3

3.1 The limit of a uniform convergence is of rank one

We should call a point, a curve and a surface in R^3 to be of rank 1, rank 2 and rank 3 respectively. We shall discuss how the limit of a uniform convergence of an iterated sequence that will result in a point, a curve and a surface in R^3.

Theorem 12 *Let S be a given closed surface $[x_0(u_1,u_2), y_0(u_1,u_2), z_0(u_1,u_2)]$, and the point $A = (p_1, q_1, w_1)$ is fixed and is not on the surface S. For r_1 and r_2 being two distinct real numbers in $(0,1)$, we let*

$$\overrightarrow{OM_1} = \begin{bmatrix} x_1(u_1,u_2) \\ y_1(u_1,u_2) \\ z_1(u_1,u_2) \end{bmatrix} = \frac{r_1 \cdot OQ}{r_1 OQ + r_2 OA}\overrightarrow{OA} + \frac{r_2 \cdot OA}{r_1 OQ + r_2 OA}\overrightarrow{OQ},$$

where Q is a moving point on S, and the locus M_1 is described in $(x_1(u_1,u_2), y_1(u_1,u_2), z_1(u_1,u_2))$. Now for $n \in \mathbb{Z}^+$, we consider

$$\overrightarrow{OM_{n+1}} = \begin{bmatrix} x_{n+1}(u_1,u_2) \\ y_{n+1}(u_1,u_2) \\ z_{n+1}(u_1,u_2) \end{bmatrix} = \frac{r_1 \cdot OQ_n}{r_1 OQ_n + r_2 OA}\overrightarrow{OA} + \frac{r_2 \cdot OA}{r_1 OQ_n + r_2 OA}\begin{bmatrix} x_n(u_1,u_2) \\ y_n(u_1,u_2) \\ z_n(u_1,u_2) \end{bmatrix},$$

where Q_n is a moving point on $[x_n(u_1,u_2), y_n(u_1,u_2), z_n(u_1,u_2)]$. Then $\overrightarrow{OM_n(u_1,u_2)} \to \overrightarrow{OA}$ as $n \to \infty$ uniformly, $\overrightarrow{M_n(u_1,u_2)\,M_{n-1}(u_1,u_2)}$ converges uniformly to 0, and $\{M_n(u_1,u_2)\}_{n=1}^{\infty}$ converges uniformly to the point A for all for all $(u_1,u_2) \in [0,2\pi] \times [0,2\pi]$.

Proof: The convergence of $\{M_n(u_1,u_2)\}_{n=1}^{\infty}$ follows directly from the corresponding 2D Theorem (5), which we omit here. ■

Example 13 *Let S be the given closed surface*

$$[x_0(u_1,u_2), y_0(u_1,u_2), z_0(u_1,u_2)] = [5\cos(u_1)\sin(u_2), 4\sin(u_1)\sin(u_2), 3\cos(u_2)],$$

and the point $A=(1,2,3)$ be fixed. For r_1 and $r_2 \in (0,1)$, and

$$\overrightarrow{OM_{n+1}} = \begin{bmatrix} x_{n+1}(u_1,u_2) \\ y_{n+1}(u_1,u_2) \\ z_{n+1}(u_1,u_2) \end{bmatrix} = \frac{r_1 \cdot OQ_n}{r_1 OQ_n + r_2 OA}\overrightarrow{OA} + \frac{r_2 \cdot OA}{r_1 OQ_n + r_2 OA}\begin{bmatrix} x_n(u_1,u_2) \\ y_n(u_1,u_2) \\ z_n(u_1,u_2) \end{bmatrix}.$$

Then $\{M_n(u_1,u_2)\}_{n=1}^{\infty}$ converges uniformly to the point A.

We depict the convergence for $r_1=\frac{1}{3}$ and $r_2=\frac{2}{3}$, and the plots of $\left\{\overrightarrow{OM_2}, \overrightarrow{OM_3}, \overrightarrow{OM_4}, \overrightarrow{OM_5}\right\}$ and the point $A=(1,2,3)$ in Figure 8:

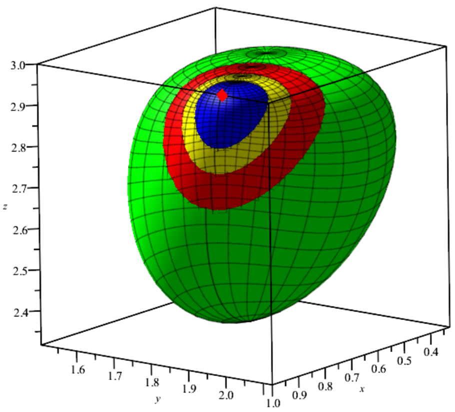


Figure 8. 3D convergence to a point.

It is natural to observe that the uniform convergence of $\{M_n(u_1,u_2)\}_{n=1}^{\infty}$ to the point A will be invariant when starting with difference surfaces, which we demonstrate this using difference closed surfaces next.

Example 14 *If we replace S to be the closed surface of $S_2 = [\cos(u_1)\sin(u_2), \sin(u_1)\cos(u_2), \cos(u_2)+1]$, and the point $A=(1,2,3)$ be fixed. Furthermore, we pick $r_1=\frac{1}{3}$, and $r_2=\frac{2}{3}$, we depict the nested plots of $\{M_2(u_1,u_2), M_3(u_1,u_2), M_4(u_1,u_2), M_5(u_1,u_2)\}$ and the point $A=(1,2,3)$ below on Figure 9(a). The plot of $M_5(u_1,u_2)$ and the point A (shown in red) is depicted in the Figure 9(b). We also plot the Figure 8 together with Figure 9(a) in Figure 9(c) below, which* *we can see both sequences of closed surfaces converge to the same point A.*

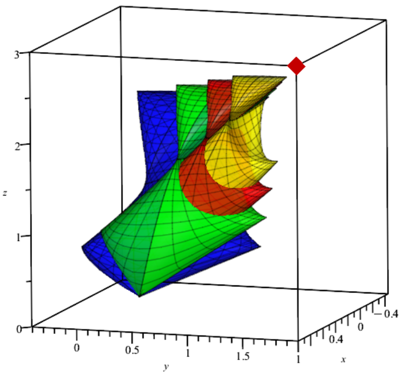


Figure 9(a). Sequence of surfaces converge to the point A.

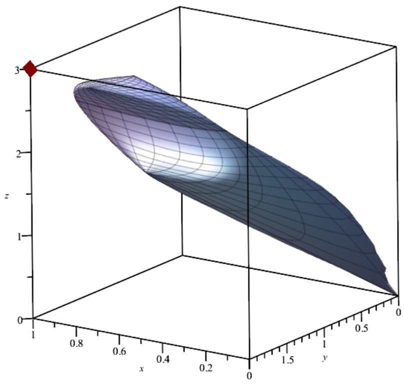


Figure 9(b). The plots of $M_5\,(u_1,u_2)$ and A.

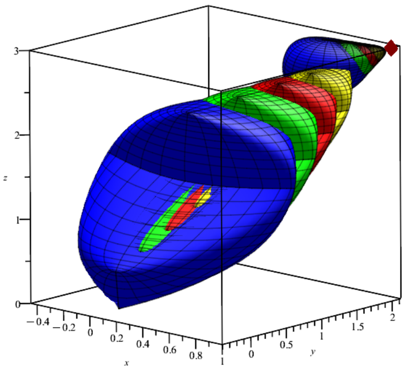


Figure 9(c). Convergences do not depend on the original surface C.

Exercise: If we use the same point A, and same coefficients $r_1 = \frac{1}{3}$ and $r_2 = \frac{2}{3}$, but use the surface S_3 of $\begin{bmatrix} 2\cos(u_1)\sin(u_1)\cos(u_1)\sin(u_2)+1 \\ 2\cos(u_1)\sin(u_1)\sin(u_1)\sin(u_2)+2 \\ 2\cos(u_1)\sin(u_1)\cos(u_2)-3 \end{bmatrix}$ as expected, we should see another sequence of surfaces converge uniformly to the same point A (shown in red in Figure 10).

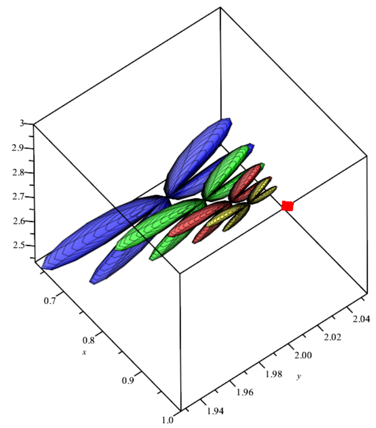


Figure 10. Convergence of S_3 and A.

3.2 The limit of a uniform convergence is of rank two

Now we consider the locus of two moving vectors and with two fixed vectors in $\mathbb{R}^3$.

Theorem 15 *Let C be a given closed non-zero surface $[x_0(u,v), y_0(u,v), z_0\,(u,v)]$, $A = (p_1, q_1, w_1)$, $B = (p_2, q_2, w_2)$ be two distinct points. Let D be the space curve lying on the surface of $(x_2(u,v), y_2(u,v), z_2(u,v))$*

when either u_1 or u_2 being kept as a constant. Suppose $v = v_0$, for r_1, r_2, r_3 and $r_4 \in (0,1)$, we let

$$\begin{aligned}\overrightarrow{OM_1} &= \begin{bmatrix} x_1(u,v) \\ y_1(u,v) \\ z_1(u,v) \end{bmatrix} = \frac{r_1 \cdot OQ}{r_2OA + r_3OB + r_1OQ + r_4OD}\overrightarrow{OA} + \frac{r_2 \cdot OA}{r_2OA + r_3OB + r_1OQ + r_4OD}\overrightarrow{OB} \\ &+ \frac{r_3 \cdot OB}{r_2OA + r_3OB + r_1OQ + r_4OD}\overrightarrow{OD\,(u, v_0)} + \frac{r_4 \cdot OC}{r_2OA + r_3OB + r_1OQ + r_4OD}\overrightarrow{OQ},\end{aligned}$$

where Q is a moving point on C, and the locus M_1 is described in $(x_1(u,v), y_1(u,v), z_1(u,v))$. Now for $n \in \mathbb{Z}^+$, if

$$\begin{aligned}\overrightarrow{OM_n} &= \begin{bmatrix} x_n(u,v) \\ y_n(u,v) \\ z_n(u,v) \end{bmatrix} \\ &= \frac{r_1 \cdot OQ_n}{r_2OA + r_3OB + r_1OQ_n + r_4OD}\overrightarrow{OA} + \frac{r_2 \cdot OA}{r_2OA + r_3OB + r_1OQ_n + r_4OD}\overrightarrow{OB} \\ &+ \frac{r_3 \cdot OB}{r_2OA + r_3OB + r_1OQ_n + r_4OD}\overrightarrow{OD\,(u, v_0)} \\ &+ \frac{r_4 \cdot OC}{r_2OA + r_3OB + r_1OQ_n + r_4OD}\begin{bmatrix} x_{n-1}(u,v) \\ y_{n-1}(u,v) \\ z_{n-1}(u,v) \end{bmatrix}, \end{aligned} \tag{17}$$

where Q_n is a moving point on $[x_n(u,v), y_n(u,v), z_n(u,v)]$. If $\overrightarrow{M_n\,(u,v)}$ converges, then $\overrightarrow{M_n\,(u,v)}$ converges uniformly to a space curve spanned by $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OD\,(u,v_0)}$, where $u \in [0, 2\pi]$.

Proof: The proof is standard which we omit here.

Now we consider a 3D Locus of three moving vectors and one fixed vector as follows, which we leave the proof to the readers.

3.3 The limit of a uniform convergence is of full rank

Now, we consider a scenario when the limit of a uniform convergence $\{M_n\,(u,v)\}$ is another two variables 3D surface.

Theorem 16 *Let C be a given closed non-zero surface $[x_0(u,v), y_0(u,v), z_0\,(u,v)]$. Let D, E and F be three distinct surfaces of $(x_2(u,v), y_2(u,v), z_2(u,v))$, $(x_3(u,v), y_3(u,v), z_3(u,v))$, and $(x_4(u,v), y_4(u,v), z_4(u,v))$ respectively. For r_1, r_2, r_3 and $r_4 \in (0,1)$, we let*

$$\begin{aligned}\overrightarrow{OM_1} &= \begin{bmatrix} x_1(u,v) \\ y_1(u,v) \\ z_1(u,v) \end{bmatrix} = \frac{r_1 \cdot OQ}{r_2OF + r_3OE + r_1OQ + r_4OD}\overrightarrow{OF\,(u,v)} \\ &+ \frac{r_2 \cdot OF}{r_2OF + r_3OE + r_1OQ + r_4OD}\overrightarrow{OE\,(u,v)} \\ &+ \frac{r_3 \cdot OE}{r_2OF + r_3OE + r_1OQ + r_4OD}\overrightarrow{OD\,(u,v)} \\ &+ \frac{r_4 \cdot OD}{r_2OF + r_3OE + r_1OQ + r_4OD}\overrightarrow{OQ},\end{aligned}$$

where Q is a moving point on C, and the locus M_1 is described in $(x_1(u,v), y_1(u,v), z_1(u,v))$. Now for $n \in \mathbb{Z}^+$, if

$$\begin{aligned}
\overrightarrow{OM_n} &= \begin{bmatrix} x_n(u,v) \\ y_n(u,v) \\ z_n(u,v) \end{bmatrix} \\
&= \frac{r_1 \cdot OQ_n}{r_2OF + r_3OE + r_1OQ_n + r_4OD}\overrightarrow{OF(u,v)} + \frac{r_2 \cdot OF}{r_2OF + r_3OE + r_1OQ_n + r_4OD}\overrightarrow{OE(u,v)} \\
&\quad + \frac{r_3 \cdot OE}{r_2OF + r_3OE + r_1OQ_n + r_4OD}\overrightarrow{OD(u,v)} \\
&\quad + \frac{r_4 \cdot OD}{r_2OF + r_3OE + r_1OQ_n + r_4OD}\begin{bmatrix} x_{n-1}(u,v) \\ y_{n-1}(u,v) \\ z_{n-1}(u,v) \end{bmatrix},
\end{aligned}$$

where Q_n is a moving point on $[x_n(u,v), y_n(u,v), z_n(u,v)]$, and if $\overrightarrow{M_n(u,v)}$ converges, then $\overrightarrow{M_n(u,v)}$ converges uniformly to the surface that is generated by $\overrightarrow{OD(u,v)}, \overrightarrow{OE(u,v)}$ and $\overrightarrow{OF(u,v)}$, where $u \in [0, 2\pi]$, and $v \in [0, \pi]$.

Proof: We assume $\begin{bmatrix} x_{n+1}(u,v) \\ y_{n+1}(u,v) \\ z_{n+1}(u,v) \end{bmatrix} \to F^* = \begin{bmatrix} p(u,v) \\ q(u,v) \\ w(u,v) \end{bmatrix}$, we denote it as $\begin{bmatrix} p \\ q \\ w \end{bmatrix}$ in brevity, then the norm of the vector, $\left\| \begin{bmatrix} x_{n+1}(u,v) \\ y_{n+1}(u,v) \\ z_{n+1}(u,v) \end{bmatrix} \right\|$, converges to $\|F^*\| = \sqrt{p^2+q^2+w^2}$, and we have

$$\begin{aligned}
&\left(1 - \frac{r_4 \cdot OD}{r_2OF + r_3OE + r_1\|F^*\| + r_4OD}\right)\begin{bmatrix} p \\ q \\ w \end{bmatrix} \\
&= \frac{r_1\|F\|}{r_2OF + r_3OE + r_1\|F^*\| + r_4OD}\overrightarrow{OF} \\
&\quad + \frac{r_2OF}{r_2OF + r_3OE + r_1\|F^*\| + r_4OD}\overrightarrow{OE} \\
&\quad + \frac{r_3 \cdot OE}{r_2OF + r_3OE + r_1\|F^*\| + r_4OD}\overrightarrow{OD}
\end{aligned}$$

$$\begin{aligned}
&\begin{bmatrix} p \\ q \\ w \end{bmatrix} \\
&= \left(\frac{r_2OF + r_3OE + r_1\|F^*\| + r_4OD}{r_2OF + r_3OE + r_1\|F^*\|}\right)\left(\begin{array}{c} \frac{r_1\|F^*\|}{r_2OF+r_3OE+r_1\|F^*\|+r_4OD}\overrightarrow{OF} \\ +\frac{r_2OF}{r_2OF+r_3OE+r_1\|F^*\|+r_4OD}\overrightarrow{OE} + \frac{r_3 \cdot OE}{r_2OF+r_3OE+r_1\|F^*\|+r_4OD}\overrightarrow{OD} \end{array}\right) \\
&= \left(\frac{r_1\|F^*\|}{r_2OF + r_3OE + r_1\|F^*\|}\right)\overrightarrow{OF} + \left(\frac{r_2OF}{r_2OF + r_3OE + r_1\|F^*\|}\right)\overrightarrow{OE} \\
&\quad + \left(\frac{r_3 \cdot OE}{r_2OF + r_3OE + r_1\|F^*\|}\right)\overrightarrow{OD}. \qquad (18)
\end{aligned}$$

4 Conclusions

We first remark that there are many other areas that readers can extend from this paper. For example, we assumed that in several places that the sequence $\overrightarrow{M_n(u,v)}$ converges first before we state the conclusions, one can make use of Cauchy Criterion for uniform convergence to search for sufficient conditions for a sequence $\overrightarrow{M_n(u,v)}$ being convergent. In addition, we can extend the plots of convex combinations to the plots of conical combinations both in 2D and 3D. Finally, one can talk about the applications of the uniform convergence of sequences of parametric equations, and infinite series of parametric equations. Nevertheless, in this paper, we have seen some interesting graphics that resulted from sequence of convex combinations of vectors in 2D and 3D. Also, readers should have gained some insights how we can comprehend a complex concept of uniform convergence of sequences of parametric curves or surfaces. As a reminder, we indeed extended a simple college exam practice problem on locus into various interesting exploratory activities, both in 2D and 3D settings. Consequently, these exploratory activities have led to many interesting areas of computer graphics by integrating mathematical knowledge in Multivariable Calculus, Advanced Calculus, and Linear Algebra. We thus propose that a math curriculum should include proper components of exploration with the help of technological tools, especially where real life applications can be found.

It is common sense that teaching to a test can never promote creative thinking skills, it could even lose potential students who might pursue mathematics related fields in the future. We know that addressing the importance and timely adoption of technological tools in teaching, learning and research can never be wrong. Access to technological tools has motivated us to rethink how mathematics can and should be presented more interestingly and also how mathematics can be made a more cross disciplinary subject. There is no doubt that evolving technological tools have helped learners to discover mathematics and to become aware of its applications.

5 Acknowledgements

Author would like to express sincere thanks to Harald Pleym of Norway for writing sequence of plots using Maple [1].

6 Supplementary Electronic Materials

[S1] A Maple file for Example 11:

$$https://atcm.mathandtech.org/EP2023/invited/22003/ATCM2023.mw$$

References

[1] Maple: A product of Maplesoft, see http://maplesoft.com/.

[2] Convex combination: https://en.wikipedia.org/wiki/Convex_combination.

[3] https://en.wikipedia.org/wiki/Homothety

[4] Yang, W.-C. Locus, Parametric Equations and Innovative Use of Technological Tools (pp. page 120-133). Proceedings of the 21st ATCM, the electronic copy can be found at this URL: https://atcm.mathandtech.org/EP2016/invited/4052016_21300.pdf, ISBN:978-0-9821164-9-4 (hard copy), ISSN 1940-4204 (online version), Mathematics and Technology LLC.

[5] Yang, W.-C. From Static Locus Problems to Exploring Mathematics with Technological Tools (pp. page 67-88). The Electronic Journal of Mathematics and Technology, Volume 11, Number 2, ISSN 1933-2823, Mathematics and Technology LLC.

[6] Yang, W.-C. Graphs of Uniform Convengence on Iteration of Loci generated by Special Convex Combinations of Curves and Surfaces, the Electronnic Proceedings of the 28th Asian Technology Conference in Mathematics (ATCM 2023), ISSN 1940-4204, soft copy can be obtained from http://atcm.mathandtech.org/EP2023/invited/22003.pdf, Mathematics and Technology LLC.